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# An Application of Medial Limits to Iterative Functional Equations, II

Janusz Morawiec 

**Abstract.** Applying medial limits we describe bounded solutions  $\varphi: S \rightarrow \mathbb{R}$  of the functional equation

$$\varphi(x) = \int_{\Omega} g(\omega)\varphi(f(x, \omega))d\mu(\omega) + G(x),$$

where  $(\Omega, \mathcal{A}, \mu)$  is a measure space,  $S \subset \mathbb{R}$ ,  $f: S \times \Omega \rightarrow S$ ,  $g: \Omega \rightarrow \mathbb{R}$  is integrable and  $G: S \rightarrow \mathbb{R}$  is bounded. The main purpose of this paper is to extend results obtained in Morawiec (Results Math 75(3):102, 2020) to the above general functional equation in wider classes of functions and under weaker assumptions.

**Mathematics Subject Classification.** Primary 39B22, 46B15; Secondary 28A20, 46B45.

**Keywords.** Banach limits, medial limits, iterative functional equations, bounded solutions.

## 1. Introduction

Throughout this paper we assume that  $(\Omega, \mathcal{A}, \mu)$  is a measure space,  $S \subset \mathbb{R}$  is a non-empty set,  $f: S \times \Omega \rightarrow S$  is a function,  $G: S \rightarrow \mathbb{R}$  is a bounded function and  $g: \Omega \rightarrow \mathbb{R}$  is an integrable function, i.e.

$$\int_{\Omega} |g(\omega)|d\mu(\omega) < \infty. \quad (1)$$

We are interested in solutions  $\varphi \in B(S, \mathbb{R})$  of the following iterative functional equation

$$\varphi(x) = \int_{\Omega} g(\omega)\varphi(f(x, \omega))d\mu(\omega) + G(x), \quad (\text{E}_G)$$

where  $B(S, \mathbb{R})$  denotes the space of all bounded functions from  $S$  to  $\mathbb{R}$  endowed with the supremum norm. The main theorems of this paper generalize results obtained in [14] in three directions; to more general functional equations in wider classes of functions and under weaker assumptions.

Functional equation  $(E_G)$ , as well as its generalizations and special cases, are investigated in various classes of functions in connection with their appearance in miscellaneous fields of science (for more details see [7, Chapter XIII], [8, Chapters 6, 7] and [2, Section 4]). Iteration is the fundamental technique for solving functional equations in a single variable, and in most cases the formulae for solutions is obtained by taking the limit of sequences in which iterates are involved. In this paper we make use of this fundamental technique, but following the ideas of [14] we apply a subclass of Banach limits instead of the limit. The idea of replacing the limit by a Banach limit seems to be clear, because we do not need any additional assumption guaranteeing the existence of a Banach limit of a bounded sequence.

We need to integrate the pointwise Banach limit of a bounded sequence of measurable functions. However, the problem is that there is no guarantee that the pointwise Banach limit of a bounded sequence of measurable functions is a measurable function (see [17, page 288]). Fortunately, it is known that there are Banach limits, called medial limits, possessing exactly the required property (see [12]; cf. [11]).

The paper is organized as follows. In Sect. 2 we set up notation and terminology. We also introduce an important operator that play the key role in describing bounded solutions of equation  $(E_G)$ . Next we start our investigation of the introduced operator. A quick look on it in the simplest contractive case is contained in Sect. 3. Next, in Sect. 4 we proceed with the detailed analysis of the operator in the non-expansive case. This section, the main part of this paper, contains generalizations of all results from [14] without any assumptions on the function  $f$ . In Sect. 5 we say a few words about the operator in the expansive case. Finally, Sect. 6 contains extensions of the main results from [14] under a bounded condition on the function  $f$ .

## 2. Preliminaries

Put

$$\mathcal{M}(S, \mathbb{R}) = \{h \in B(S, \mathbb{R}) \mid h \circ f(x, \cdot) \text{ is } \mathcal{A}\text{-measurable for every } x \in S\}$$

Note that  $\mathcal{M}(S, \mathbb{R})$  is a non-trivial vector space; indeed every constant function belongs to  $\mathcal{M}(S, \mathbb{R})$ . Observe also that if  $\varphi \in \mathcal{M}(S, \mathbb{R})$ , then the integral in  $(E_G)$  is well defined, and moreover, finite by (1). Therefore, considering solutions of equation  $(E_G)$  in  $\mathcal{M}(S, \mathbb{R})$  is justified. In the case where

$\int_{\Omega} g(\omega) d\mu(\omega) = 1$  every constant function satisfies the homogeneous counterpart of (E<sub>G</sub>), i.e. equation of the form

$$\Phi(x) = \int_{\Omega} g(\omega) \Phi(f(x, \omega)) d\mu(\omega). \quad (\text{E}_0)$$

But, in the general case it is not easy to find a non-trivial solution of (E<sub>0</sub>) in  $\mathcal{M}(S, \mathbb{R})$ . Moreover, it is unclear whether equation (E<sub>G</sub>) has a solution in  $\mathcal{M}(S, \mathbb{R})$ . The problem is that we do not have tools to solve this problem in the full generality. However, some results can be proved with the use of the following observation.

**Proposition 2.1.** *The space  $\mathcal{M}(S, \mathbb{R})$  endowed with the supremum norm is complete.*

Define an operator  $\mathbf{T}: \mathcal{M}(S, \mathbb{R}) \rightarrow B(S, \mathbb{R})$  putting

$$\mathbf{T}h(x) = \int_{\Omega} g(\omega) h(f(x, \omega)) d\mu(\omega).$$

Obviously,  $\mathbf{T}$  is linear and continuous with  $\|\mathbf{T}\| \leq \int_{\Omega} |g(\omega)| d\mu(\omega)$ , by (1). Moreover, equations (E<sub>G</sub>) and (E<sub>0</sub>) can be written now in the following forms

$$\varphi = \mathbf{T}\varphi + G \quad (2)$$

and

$$\Phi = \mathbf{T}\Phi, \quad (3)$$

respectively.

From now on we fix a non-trivial subspace  $\mathcal{B}(S, \mathbb{R})$  of  $\mathcal{M}(S, \mathbb{R})$  that is invariant under  $\mathbf{T}$ , i.e.

$$\mathbf{T}(\mathcal{B}(S, \mathbb{R})) \subset \mathcal{B}(S, \mathbb{R}). \quad (4)$$

Before we give examples showing how the space  $\mathcal{B}(S, \mathbb{R})$  looks like in a certain situation, let us introduce symbols, which we will use for basic spaces of functions. Here and subsequently,  $C_A(S, \mathbb{R})$ ,  $Lip(S, \mathbb{R})$ ,  $BV(S, \mathbb{R})$  and  $Borel(S, \mathbb{R})$  denote subspaces of the space  $B(S, \mathbb{R})$  consisting of all functions that are continuous at every point of a set  $A \subset S$ , Lipschitzian, of bounded variation (in general  $S$  may not be a compact set, and in such a case by a function with bounded variation we mean a function which can be written as a difference of two increasing functions, which is justified by [10, Theorem 1.4.1]) and Borel measurable, respectively. Here and throughout this paper, increasing functions are always understood in a weak sense.

*Example 2.1.* (cf. [14, Examples 3.1–3.5])

- (i) Assume that  $\mathcal{A} = 2^{\Omega}$ . Then (4) holds with  $\mathcal{B}(S, \mathbb{R}) = B(S, \mathbb{R})$ . Moreover, if  $A \subset S$  is such that  $f(A, \Omega) \subset A$  and for every  $\omega \in \Omega$  the function  $f(\cdot, \omega)$  is continuous at every point of  $A$ , then (4) holds with  $\mathcal{B}(S, \mathbb{R}) = C_A(S, \mathbb{R})$

- (ii) If  $f$  is  $\mathcal{A}$ -measurable with respect to the second variable and increasing with respect to the first variable in the case where  $g \geq 0$  or decreasing with respect to the first variable in the case where  $g \leq 0$ , then (4) holds with  $\mathcal{B}(S, \mathbb{R}) = BV(S, \mathbb{R})$ .
- (iii) If  $f$  is  $\mathcal{A}$ -measurable with respect to the second variable and there exists  $\alpha > 0$  such that  $\int_{\Omega} |g(\omega)| |f(x, \omega) - f(y, \omega)| d\mu(\omega) \leq \alpha |x - y|$  for all  $x, y \in S$ , then (4) holds with  $\mathcal{B}(S, \mathbb{R}) = Lip(S, \mathbb{R})$ .
- (iv) Assume that  $S$  is a Borel set and  $f$  is a measurable with respect to  $\sigma$ -algebra  $\mathcal{B}(S) \times \mathcal{A}$ , where  $\mathcal{B}(S)$  is the  $\sigma$ -algebra of all Borel subsets of  $S$ . Then (4) holds with  $\mathcal{B}(S, \mathbb{R}) = Borel(S, \mathbb{R})$ . Moreover, if  $A \subset S$ ,  $f(A, \Omega) \subset A$  and for every  $\omega \in \Omega$  the function  $f(\cdot, \omega)$  is continuous at every point of  $A$ , then (4) holds with  $\mathcal{B}(S, \mathbb{R}) = C_A(S, \mathbb{R}) \cap Borel(S, \mathbb{R})$ .

For simplifying statements of our results let us denote by  $\mathfrak{sol}(E_0)$  and  $\mathfrak{sol}(E_G)$  families of all functions from the space  $\mathcal{B}(S, \mathbb{R})$  satisfying equation (E<sub>0</sub>) and (E<sub>G</sub>), respectively, i.e.  $\mathfrak{sol}(E_0) = \{\Phi \in \mathcal{B}(S, \mathbb{R}) \mid \Phi = \mathbf{T}\Phi\}$  and  $\mathfrak{sol}(E_G) = \{\varphi \in \mathcal{B}(S, \mathbb{R}) \mid \varphi = \mathbf{T}\varphi + G\}$ .

In later sections we will need the medial limits. To introduce them denote by  $\mathfrak{B}$  the family of all Banach limits defined on  $B(\mathbb{N}, \mathbb{R})$ . Recall that  $M \in \mathfrak{B}$  if  $M: B(\mathbb{N}, \mathbb{R}) \rightarrow \mathbb{R}$  is a linear, positive, shift invariant and normalized operator. It is easy to see that any  $M \in \mathfrak{B}$  is continuous with  $\|M\| = 1$ . Let us note that the cardinality of  $\mathfrak{B}$  is equal to  $2^{\mathfrak{c}}$  (see [5]; cf. [3], where it is proved that the cardinality of the set of all extreme points of  $\mathfrak{B}$  is equal to  $2^{\mathfrak{c}}$ ); here  $\mathfrak{c}$  is the cardinality of the continuum. A Banach limit  $M$  is called a medial limit, with respect to a probability space  $(\Omega, \mathcal{A}, P)$ , if  $\int_{\Omega} M((h_m(\omega))_{m \in \mathbb{N}}) dP(\omega)$  is defined and equal to  $M((\int_{\Omega} h_m(\omega) dP(\omega))_{m \in \mathbb{N}})$  whenever  $(h_m)_{m \in \mathbb{N}}$  is a bounded sequence of measurable functions from  $\Omega$  to  $\mathbb{R}$ ; the sentence  $\int_{\Omega} M((h_m(\omega))_{m \in \mathbb{N}}) dP(\omega)$  is defined means, in particular, that the function  $M((h_m(\cdot))_{m \in \mathbb{N}})$  is  $\mathcal{A}$ -measurable. It is known that the continuum hypothesis implies the existence of medial limits. More results on the existence and non-existence of medial limits can be found in [6, Chapter 53] and in [9]. Denote by  $\mathfrak{M}_P$  the family of all medial limits, with respect to a probability space  $(\Omega, \mathcal{A}, P)$ , i.e.  $M \in \mathfrak{M}_P$  if  $M \in \mathfrak{B}$  and

$$\int_{\Omega} M((h_m(\omega))_{m \in \mathbb{N}}) dP(\omega) = M\left(\left(\int_{\Omega} h_m(\omega) dP(\omega)\right)_{m \in \mathbb{N}}\right)$$

for every sequence  $(h_m)_{m \in \mathbb{N}}$  of bounded measurable real-valued functions defined on  $\Omega$ . Note that  $\mathfrak{M}_P = \mathfrak{B}$  in the case where  $\mathcal{A} = 2^{\Omega}$ .

### 3. The Case Where $\mathbf{T}$ is Contractive

We begin with two simple results on equation (E<sub>G</sub>). The first one concerns the uniqueness of solutions in the space  $\mathcal{M}(S, \mathbb{R})$ , whereas in the second one we get the existence of a solution in the space  $\mathcal{B}(S, \mathbb{R})$ .

**Proposition 3.1.** Assume that  $G \in \mathcal{M}(S, \mathbb{R})$  and that

$$\|\mathbf{T}\| < 1. \quad (5)$$

Then equation  $(E_G)$  has at most one solution in the space  $\mathcal{M}(S, \mathbb{R})$ . In particular, equation  $(E_0)$  has no non-trivial solution in the space  $\mathcal{M}(S, \mathbb{R})$ .

*Proof.* Fix  $\varphi_1, \varphi_2 \in \mathcal{M}(S, \mathbb{R})$  satisfying  $(E_G)$ . Then  $\Phi = \varphi_1 - \varphi_2 \in \mathcal{M}(S, \mathbb{R})$  satisfies  $(E_0)$ . Hence  $\|\Phi\| = \|\mathbf{T}\Phi\| \leq \|\mathbf{T}\|\|\Phi\|$ . This jointly with (5) yields  $\|\Phi\| = 0$ , and therefore  $\varphi_1 = \varphi_2$ .  $\square$

Before we formulate the second result let us note that  $G \in \mathcal{B}(S, \mathbb{R})$  is a necessary condition for equation  $(E_G)$  to have a solution in  $\mathcal{B}(S, \mathbb{R})$ , by (4).

**Proposition 3.2.** Assume that  $G \in \mathcal{B}(S, \mathbb{R})$  and that (5) holds. If  $\mathcal{B}(S, \mathbb{R})$  is closed, then  $\text{sol}(E_G) = \{\varphi_0\}$ , where  $\varphi_0 = \sum_{m \in \mathbb{N}_0} \mathbf{T}^m G$ .

*Proof.* According to Proposition 2.1 we conclude that the space  $\mathcal{B}(S, \mathbb{R})$  is complete. By (5) the operator  $P: \mathcal{B}(S, \mathbb{R}) \rightarrow \mathcal{B}(S, \mathbb{R})$  given by  $Ph = \mathbf{T}h + G$  is a contraction. Thus the Banach fixed point theorem implies that  $\text{sol}(E_G) = \{\varphi_0\}$ . To get the formula for  $\varphi_0$  we first note that (5) yields convergence of the series  $\sum_{m \in \mathbb{N}_0} \mathbf{T}^m G$ . From (4) and the fact that  $\mathcal{B}(S, \mathbb{R})$  is closed we have  $\sum_{m \in \mathbb{N}_0} \mathbf{T}^m G \in \mathcal{B}(S, \mathbb{R})$ . Finally,  $\mathbf{T}(\sum_{m \in \mathbb{N}_0} \mathbf{T}^m G) + G = \sum_{m \in \mathbb{N}} \mathbf{T}^m G + G = \sum_{m \in \mathbb{N}_0} \mathbf{T}^m G$ .  $\square$

In general, it can happen that equation  $(E_G)$  has a solution in the space  $\mathcal{M}(S, \mathbb{R})$ , which is not of the form  $\sum_{m \in \mathbb{N}_0} \mathbf{T}^m G$  (see [15]).

Combining Proposition 3.2 with Example 2.1 (i) and (iv) we obtain the following corollaries.

**Corollary 3.1.** Assume that  $(p_n)_{n \in \mathbb{N}}$  is a sequence of real numbers and  $(f_n)_{n \in \mathbb{N}}$  is a sequence of self-mappings of  $S$ . If  $\sum_{n \in \mathbb{N}} |p_n| < 1$ , then the equation

$$\varphi(x) = \sum_{n \in \mathbb{N}} \varphi(f_n(x)) + G(x),$$

has exactly one bounded solution  $\varphi \in \mathcal{B}(S, \mathbb{R})$ ; it is given by the formula

$$\varphi(x) = G(x) + \sum_{k \in \mathbb{N}} \sum_{n_1, \dots, n_k \in \mathbb{N}} p_{n_1} \cdots p_{n_k} G(f_{n_1} \circ \dots \circ f_{n_k}(x)).$$

**Corollary 3.2.** Assume that  $S$  is a Borel set,  $f$  is a measurable with respect to  $\sigma$ -algebra  $\mathcal{B}(S) \times \mathcal{A}$  and  $G$  is Borel measurable. If (5) holds, then equation  $(E_G)$  has exactly one solution  $\varphi \in \text{Borel}(S, \mathbb{R})$ ; it is given by the formula

$$\varphi(x) = G(x) + \sum_{k=1}^{\infty} \int_{\Omega^k} \prod_{l=1}^k g(\omega_l) G(f^{[k]}(x, \omega_1, \dots, \omega_k)) d\mu(\omega_1, \dots, \omega_k),$$

where  $f^{[1]} = f$  and  $f^{[k+1]}(x, \omega_1, \dots, \omega_k, \omega_{k+1}) = f(f^{[k]}(x, \omega_1, \dots, \omega_k), \omega_{k+1})$  for all  $x \in S$ ,  $(\omega_1, \dots, \omega_{k+1}) \in \Omega^{k+1}$  and  $k \in \mathbb{N}$ .

Moreover, if  $A \subset S$  is such that  $f(A, \Omega) \subset A$ ,  $f(\cdot, \omega)$  is continuous at every point of  $A$  for any  $\omega \in \Omega$  and  $G$  is continuous at every point of  $A$ , then  $\varphi$  is continuous at every point of  $A$ .

#### 4. The Case Where $\mathbf{T}$ is Non-expansive

The aim of this section is to generalize results obtained in [14] without any assumptions on the function  $f$ .

Assume that

$$\|\mathbf{T}\| = 1. \quad (6)$$

First of all let us note that all results of this section holds true in the case where  $\|\mathbf{T}\| \leq 1$ , whereas Propositions 3.1 and 3.2 can be applied only if  $\|\mathbf{T}\| \leq 1$ .

By (6) for every  $h \in \mathcal{B}(S, \mathbb{R})$  we have  $\sup_{m \in \mathbb{N}} \|\mathbf{T}^m h\| \leq \|h\|$ , and hence  $(\mathbf{T}^m h(x))_{m \in \mathbb{N}} \in B(\mathbb{N}, \mathbb{R})$  for every  $x \in S$ . Therefore, given a function  $h \in \mathcal{B}(S, \mathbb{R})$  and a Banach limit  $M \in \mathfrak{B}$  we can associate with them a function  $M_h: S \rightarrow \mathbb{R}$  defined by

$$M_h(x) = M((\mathbf{T}^m h(x))_{m \in \mathbb{N}}).$$

The functions  $M_h$  play a key role in determining  $\mathfrak{sol}(E_0)$  and  $\mathfrak{sol}(E_G)$ . So, we need some facts about them. We begin with the following simple observation.

**Lemma 4.1.** *Assume that (6) holds. If  $M \in \mathfrak{B}$ , then*

$$\mathfrak{sol}(E_0) \subset \{M_h \mid h \in \mathcal{B}(S, \mathbb{R})\}.$$

*Proof.* Fix  $M \in \mathfrak{B}$  and  $\Phi \in \mathfrak{sol}(E_0)$ . Then  $\Phi \in \mathcal{B}(S, \mathbb{R})$  and  $M_\Phi$  is well defined by (6). Moreover, (3) yields  $\mathbf{T}^m \Phi = \Phi$  for every  $m \in \mathbb{N}$ . Hence  $\Phi(x) = M((\Phi(x))_{m \in \mathbb{N}}) = M((\mathbf{T}^m \Phi(x))_{m \in \mathbb{N}}) = M_\Phi(x)$  for every  $x \in S$ .  $\square$

Before we formulate the next lemma we introduce two probability measures that will be useful in the next proofs.

Put  $\alpha_1 = \int_\Omega \max\{g(\omega), 0\} d\mu(\omega)$  and  $\alpha_2 = -\int_\Omega \min\{g(\omega), 0\} d\mu(\omega)$ . By (6) and (1) we have

$$1 = \|\mathbf{T}\| \leq \int_\Omega |g(\omega)| d\mu(\omega) = \alpha_1 + \alpha_2 < \infty.$$

Now, define probability measures  $P_1, P_2: \mathcal{A} \rightarrow [0, 1]$  putting

$$P_1(A) = \begin{cases} \frac{1}{\alpha_1} \int_A \max\{g(\omega), 0\} d\mu(\omega), & \text{if } \alpha_1 > 0, \\ \int_A |g(\omega)| d\mu(\omega), & \text{if } \alpha_1 = 0, \end{cases}$$

$$P_2(A) = \begin{cases} \frac{1}{\alpha_2} \int_A \max\{-g(\omega), 0\} d\mu(\omega), & \text{if } \alpha_2 > 0, \\ \int_A |g(\omega)| d\mu(\omega), & \text{if } \alpha_2 = 0. \end{cases}$$

It is easy to see that for every  $A \in \mathcal{A}$  we have

$$\int_A g(\omega) d\mu(\omega) = \alpha_1 P_1(A) - \alpha_2 P_2(A). \quad (7)$$

Note that this equality can also be obtained by the Jordan decomposition of the signed measure defined by  $\nu(A) = \int_A g(\omega) d\mu(\omega)$  (see [4, Section IX.2]), but the difference is that in the case where  $P$  is of constant sing, there is formally only one measure in the Jordan distribution.

To the end of this paper we assume that

$$\mathfrak{M}_{P_1} \cap \mathfrak{M}_{P_2} \neq \emptyset, \quad (8)$$

where  $\mathfrak{M}_{P_1}$  and  $\mathfrak{M}_{P_2}$  are families of medial limits with respect to  $(\Omega, \mathcal{A}, P_1)$  and  $(\Omega, \mathcal{A}, P_2)$ , respectively. In general, there is no guarantee that (8) is satisfied. But, there are quite simple situations when (8) holds. For example, if  $\alpha_1 \cdot \alpha_2 = 0$ , then  $P_1 = P_2$ , and hence  $\mathfrak{M}_{P_1} = \mathfrak{M}_{P_2}$ . Another example is when  $\{\omega \in \Omega \mid g(\omega) > 0\}$  or  $\{\omega \in \Omega \mid g(\omega) < 0\}$  is a countable set, because then  $\mathfrak{M}_{P_1} = \mathfrak{B}$  or  $\mathfrak{M}_{P_2} = \mathfrak{B}$ , respectively.

Now, we need a counterpart of [14, Lemma 3.1].

**Lemma 4.2.** *Assume that (6) holds and that  $M \in \mathfrak{M}_{P_1} \cap \mathfrak{M}_{P_2}$ . If  $h \in \mathcal{B}(S, \mathbb{R})$ , then  $M_h \in \mathcal{M}(S, \mathbb{R})$  and  $\mathbf{T}M_h = M_h$ .*

*Proof.* Fix  $h \in \mathcal{B}(S, \mathbb{R})$ . From (4) we conclude that  $\mathbf{T}^m h \in \mathcal{B}(S, \mathbb{R})$  for every  $m \in \mathbb{N}$ . Then (6) yields

$$\sup_{x \in S} |M_h(x)| \leq \sup_{x \in S} \|M\| \sup_{m \in \mathbb{N}} |\mathbf{T}^m h(x)| \leq \|h\|,$$

which yields  $M_h \in \mathcal{B}(S, \mathbb{R})$ .

By (4) we see that the function  $\mathbf{T}^m h \circ f(x, \cdot)$  is  $\mathcal{A}$ -measurable for all  $m \in \mathbb{N}$  and  $x \in S$ . Since  $M \in \mathfrak{M}_{P_1}$ , the integral  $\int_{\Omega} M((\mathbf{T}^m h(f(x, \omega)))_{m \in \mathbb{N}}) dP_1(\omega)$  is defined. In particular, the function  $M_h \circ f(x, \cdot) = M((\mathbf{T}^m h(f(x, \cdot)))_{m \in \mathbb{N}})$  is  $\mathcal{A}$ -measurable for every  $x \in S$ , which means that  $M_h \in \mathcal{M}(S, \mathbb{R})$ .

Applying (7) jointly with properties of medial limits we get

$$\begin{aligned} \mathbf{T}M_h(x) &= \int_{\Omega} g(\omega) M((\mathbf{T}^m h(f(x, \omega)))_{m \in \mathbb{N}}) d\mu(\omega) \\ &= \alpha_1 \int_{\Omega} M((\mathbf{T}^m h(f(x, \omega)))_{m \in \mathbb{N}}) dP_1(\omega) \\ &\quad - \alpha_2 \int_{\Omega} M((\mathbf{T}^m h(f(x, \omega)))_{m \in \mathbb{N}}) dP_2(\omega) \\ &= \alpha_1 M\left(\left(\int_{\Omega} \mathbf{T}^m h(f(x, \omega)) dP_1(\omega)\right)_{m \in \mathbb{N}}\right) \\ &\quad - \alpha_2 M\left(\left(\int_{\Omega} \mathbf{T}^m h(f(x, \omega)) dP_2(\omega)\right)_{m \in \mathbb{N}}\right) \\ &= M\left(\left(\int_{\Omega} g(\omega) \mathbf{T}^m h(f(x, \omega)) d\mu(\omega)\right)_{m \in \mathbb{N}}\right) \\ &= M((\mathbf{T}^{m+1} h(x))_{m \in \mathbb{N}}) = M((\mathbf{T}^m h(x))_{m \in \mathbb{N}}) = M_h(x) \end{aligned}$$

for every  $x \in S$ . □



Now, we want to find conditions under which  $M_h \in \mathcal{B}(S, \mathbb{R})$  for every  $h \in \mathcal{B}(S, \mathbb{R})$ . Unfortunately, the situation is as in [14], i.e. there is no chance to find such conditions in the general case, because to prove that  $M_h \in \mathcal{B}(S, \mathbb{R})$ , we would have to show, by the definition of  $\mathcal{B}(S, \mathbb{R})$  (see (4)), that  $\mathbf{T}M_h \in \mathcal{B}(S, \mathbb{R})$ ; unfortunately Lemma 4.2 yields  $\mathbf{T}M_h = M_h$ . This observation suggests the following definition.

**Definition 4.1.** We say that a family  $\mathcal{F} \subset \mathcal{B}(S, \mathbb{R})$  is *closed under*  $M \in \mathfrak{B}$ , if  $M_h \in \mathcal{F}$  for every  $h \in \mathcal{F}$ .

The next example shows that there are many vector spaces that are closed under some Banach limits.

*Example 4.1.* (cf. [14, Examples 3.1–3.5])

- (i) Assume  $\mathcal{B}(S, \mathbb{R}) = B(S, \mathbb{R})$ . Then  $B(S, \mathbb{R})$  is closed under any  $M \in \mathfrak{B}$ . Moreover, if  $A \subset S$ ,  $f(A, \Omega) \subset A$  and for every  $x_0 \in A$  there exists  $\eta > 0$  such that  $\frac{f(x, \omega) - f(x_0, \omega)}{x - x_0} \leq 1$  for all  $\omega \in \Omega$  and  $x \in S$  with  $0 < |x - x_0| \leq \eta$ , then  $C_A(S, \mathbb{R})$  is closed under any  $M \in \mathfrak{B}$ .
- (ii) If  $f$  is  $\mathcal{A}$ -measurable with respect to the second variable and increasing with respect to the first variable in the case where  $g \geq 0$  or decreasing with respect to the first variable in the case where  $g \leq 0$ , then  $BV(S, \mathbb{R})$  is closed under any  $M \in \mathfrak{B}$ .
- (iii) If  $f$  is  $\mathcal{A}$ -measurable with respect to the second variable and there exists  $\alpha \leq 1$  such that  $\int_{\Omega} |g(\omega)| |f(x, \omega) - f(y, \omega)| d\mu(\omega) \leq \alpha |x - y|$  for all  $x, y \in S$ , then  $Lip(S, \mathbb{R})$  is closed under any  $M \in \mathfrak{B}$ .
- (iv) Assume that  $S$  is a Borel set and  $f$  is a measurable with respect to  $\sigma$ -algebra  $\mathcal{B}(S) \times \mathcal{A}$ . Then  $Borel(S, \mathbb{R})$  is closed under any  $M \in \mathfrak{M}_{\nu}$ , where  $\nu$  is a probability Borel measure on  $S$ .

We are now in a position to describe the family  $\mathfrak{sol}(E_0)$ . Namely, combining Lemmas 4.1 and 4.2 we obtain the following result, which generalizes [14, Theorem 3.2].

**Theorem 4.1.** Assume that (6) holds and that  $M \in \mathfrak{M}_{P_1} \cap \mathfrak{M}_{P_2}$ . If  $\mathcal{B}(S, \mathbb{R})$  is closed under  $M$ , then

$$\mathfrak{sol}(E_0) = \{M_h \mid h \in \mathcal{B}(S, \mathbb{R})\}.$$

Now we pass to describing the family  $\mathfrak{sol}(E_G)$ . To do it we need define a certain family of functions generated by  $G \in \mathcal{B}(S, \mathbb{R})$ . If  $G \in \mathcal{B}(S, \mathbb{R})$ , then (4) yields  $\{\mathbf{T}^l G \mid l \in \mathbb{N}\} \subset \mathcal{B}(S, \mathbb{R})$ . Thus for all  $G \in \mathcal{B}(S, \mathbb{R})$  and  $k \in \mathbb{N}$  we can define a function  $G_k: S \rightarrow \mathbb{R}$  by

$$G_k(x) = \sum_{l=0}^{k-1} \mathbf{T}^l G(x).$$

To the end of this section we set

$$\mathcal{G} = \{G_k \mid k \in \mathbb{N}\}$$

We begin with a necessary condition on the family  $\mathcal{G}$  for equation (E<sub>G</sub>) to have a solution in the class  $\mathcal{B}(S, \mathbb{R})$ .

**Lemma 4.3.** *Assume that (6) holds. If*

$$\mathfrak{sol}(E_G) \neq \emptyset, \quad (9)$$

*then  $\mathcal{G}$  is a bounded subset of  $\mathcal{B}(S, \mathbb{R})$ .*

*Proof.* Fix  $\varphi \in \mathfrak{sol}(E_G)$ . From (4) we see that  $\mathcal{G} \subset \mathcal{B}(S, \mathbb{R})$ . Applying induction to (2) we obtain

$$\varphi = \mathbf{T}^k \varphi + G_k \quad (10)$$

for every  $k \in \mathbb{N}$ . This jointly with (6) gives  $\sup_{k \in \mathbb{N}} \|G_k\| \leq 2\|\varphi\|$ .  $\square$

From Lemma 4.3 we see that  $M_{G_k}$  are well defined for all  $k \in \mathbb{N}$  and  $M \in \mathfrak{B}$  whenever (9) holds. So, we can ask about the formula of these functions. To answer this question fix  $\varphi \in \mathfrak{sol}(E_G)$  and  $M \in \mathfrak{B}$ . Then (6) implies that  $M_G$  is well defined and (2) gives  $M_G(x) = M((\mathbf{T}^m G(x))_{m \in \mathbb{N}}) = M((\mathbf{T}^m \varphi(x))_{m \in \mathbb{N}}) - M((\mathbf{T}^{m+1} \varphi(x))_{m \in \mathbb{N}}) = 0$  for every  $x \in S$ . Now, note that  $M_{G_k} = kM_G$  for every  $k \in \mathbb{N}$ .

If  $G \in \mathcal{B}(S, \mathbb{R})$  and if the family  $\mathcal{G}$  is bounded, then with any Banach limit  $M \in \mathfrak{B}$  we associate a function  $M_*: S \rightarrow \mathbb{R}$  defined by

$$M_*(x) = M((G_k(x))_{k \in \mathbb{N}}).$$

*Remark 4.1.* If (9) does not hold, then it may happen that there is no  $M \in \mathfrak{B}$  for which  $M_*$  is well defined. To see an example of such a situation assume that  $P$  is a probability measure and consider equation (E<sub>G</sub>) with  $f(x, \omega) = x$  and  $g(\omega) = G(x) = 1$  for all  $x \in S$  and  $\omega \in \Omega$ .

From now on we adopt the convention that  $\{M_h + M_* \mid h \in \mathcal{B}(S, \mathbb{R})\} = \emptyset$  provided that (9) does not hold.

**Lemma 4.4.** *Assume that (6) holds. If  $M \in \mathfrak{M}_{P_1} \cap \mathfrak{M}_{P_2}$ , then*

$$\mathfrak{sol}(E_G) \subset \{M_h + M_* \mid h \in \mathcal{B}(S, \mathbb{R})\}.$$

*Proof.* According to our convention we can assume that (9) holds. Fix  $\varphi \in \mathfrak{sol}(E_G)$ . Obviously,  $\varphi \in \mathcal{B}(S, \mathbb{R})$  and  $B_\varphi$  is well defined. From Lemma 4.3 we conclude that  $M_*$  is also well defined. Then making use of (10) we get  $\varphi(x) = M_\varphi(x) + M_*(x)$  for every  $x \in S$ .  $\square$

The next fact is a counterpart of [14, Lemma 4.3].

**Lemma 4.5.** *Assume that the family  $\mathcal{G}$  is bounded and that  $M \in \mathfrak{M}_{P_1} \cap \mathfrak{M}_{P_2}$ . If  $G \in \mathcal{B}(S, \mathbb{R})$ , then  $M_* \in \mathcal{M}(S, \mathbb{R})$  and  $M_* = \mathbf{T}M_* + G$ .*

*Proof.* Since  $\mathcal{G}$  is bounded,  $M_*$  is well defined and

$$\sup_{x \in S} |M_*(x)| \leq \sup_{x \in S} \|M\| \sup_{k \in \mathbb{N}} |G_k(x)| \leq \sup_{k \in \mathbb{N}} \|G_k\| < \infty.$$

Hence  $M_* \in \mathcal{B}(S, \mathbb{R})$ . Since  $M \in \mathfrak{M}_{P_1} \cap \mathfrak{M}_{P_2}$ , it follows that  $M_* \in \mathcal{M}(S, \mathbb{R})$ ; cf. the second part of the proof of Lemma 4.2. Applying (7) jointly with properties of medial limits we get

$$\begin{aligned}
 \mathbf{T}M_*(x) &= \int_{\Omega} |g(\omega)| M \left( (G_k(f(x, \omega)))_{k \in \mathbb{N}} \right) d\mu(\omega) \\
 &= \alpha_1 \int_{\Omega} M \left( (G_k(f(x, \omega)))_{k \in \mathbb{N}} \right) dP_1(\omega) \\
 &\quad - \alpha_2 \int_{\Omega} M \left( (G_k(f(x, \omega)))_{k \in \mathbb{N}} \right) dP_2(\omega) \\
 &= \alpha_1 M \left( \left( \int_{\Omega} G_k(f(x, \omega)) dP_1(\omega) \right)_{k \in \mathbb{N}} \right) \\
 &\quad - \alpha_2 M \left( \left( \int_{\Omega} G_k(f(x, \omega)) dP_2(\omega) \right)_{k \in \mathbb{N}} \right) \\
 &= M \left( \left( \int_{\Omega} g(\omega) G_k(f(x, \omega)) d\mu(\omega) \right)_{k \in \mathbb{N}} \right) \\
 &= M \left( \left( \sum_{l=0}^{k-1} \int_{\Omega} g(\omega) \mathbf{T}^l G(f(x, \omega)) d\mu(\omega) \right)_{k \in \mathbb{N}} \right) \\
 &= M \left( \left( \sum_{l=0}^{k-1} \mathbf{T}^{l+1} G(x) \right)_{k \in \mathbb{N}} \right) = M \left( \left( \sum_{l=0}^k \mathbf{T}^l G(x) - G(x) \right)_{k \in \mathbb{N}} \right) \\
 &= M \left( (G_{k+1}(x))_{k \in \mathbb{N}} \right) - G(x) = M_*(x) - G(x)
 \end{aligned}$$

for every  $x \in S$ .  $\square$

We now want to find conditions under which  $M_* \in \mathcal{B}(S, \mathbb{R})$ . The situation is similar to that for  $M_h \in \mathcal{B}(S, \mathbb{R})$ . Namely, to prove that  $M_* \in \mathcal{B}(S, \mathbb{R})$ , we would have to show that  $\mathbf{T}M_* \in \mathcal{B}(S, \mathbb{R})$ , but Lemma 4.5 yields  $\mathbf{T}M_* = M_* - G$ . This leads us to the following definition.

**Definition 4.2.** We say that a function  $G \in \mathcal{B}(S, \mathbb{R})$  is *admissible* for a Banach limit  $M \in \mathfrak{B}$ , if the family  $\mathcal{G}$  is bounded and if  $M_* \in \mathcal{B}(S, \mathbb{R})$ .

Note that the boundedness assumption on the family  $\mathcal{G}$  in the above definition is not restrictive; indeed if  $\mathcal{G}$  is unbounded, then  $M_*$  can not be a solution of equation (E<sub>G</sub>) by Lemma 4.3.

Before we give examples of admissible functions for some Banach limits, let us note the following fact.

**Lemma 4.6.** Assume that (6) holds. If (9) is satisfied, then  $G$  is admissible for each  $M \in \mathfrak{B}$  under which  $\mathcal{B}(S, \mathbb{R})$  is closed.

*Proof.* The family  $\mathcal{G}$  is bounded by Lemma 4.3. Let  $\mathcal{B}(S, \mathbb{R})$  be closed under  $M \in \mathfrak{B}$ . Fix  $\varphi \in \text{sol}(E_G)$ . Then  $M_\varphi \in \mathcal{B}(S, \mathbb{R})$ , by (6). According to (10) we conclude that  $M_* = \varphi - M_\varphi \in \mathcal{B}(S, \mathbb{R}) - \mathcal{B}(S, \mathbb{R}) = \mathcal{B}(S, \mathbb{R})$ .  $\square$

*Example 4.2.* (cf. [14, Examples 4.1–4.3])

- (i) Assume that  $\mathcal{G} \subset \mathcal{B}(S, \mathbb{R})$  and the series  $\sum_{l=0}^{\infty} \mathbf{T}^l G$  converges pointwise to a function from  $\mathcal{B}(S, \mathbb{R})$ . Then  $G$  is admissible for every  $M \in \mathfrak{B}$ . Moreover, for any  $M \in \mathfrak{B}$  we have  $M_* = \sum_{l=0}^{\infty} \mathbf{T}^l G$  and

$$M_*(x) = M((\mathbf{T}^m G_k(x))_{k \in \mathbb{N}}) + \sum_{l=0}^m \mathbf{T}^l G(x)$$

for all  $x \in S$  and  $m \in \mathbb{N}$ .

- (ii) Assume that  $\mathcal{B}(S, \mathbb{R}) = \mathcal{B}(S, \mathbb{R})$ . Then each function  $G \in \mathcal{B}(S, \mathbb{R})$  guaranteeing boundedness of  $\mathcal{G}$  is admissible for every  $M \in \mathfrak{B}$ .  
 (iii) Assume that  $G \in \mathcal{B}(S, \mathbb{R})$  and that there exists  $m \in \mathbb{N}$  such that

$$\mathbf{T}^m G = 0. \quad (11)$$

Then  $\mathcal{G} = \{ \sum_{l=0}^{k-1} \mathbf{T}^l G \mid k \in \{1, \dots, m\} \}$  and  $M_* = \sum_{l=0}^{m-1} \mathbf{T}^l G$  for any  $M \in \mathfrak{B}$ . In particular,  $G$  is admissible for any  $M \in \mathfrak{B}$ .

We now formulate the main result of this section, which jointly with Lemmas 4.4 and 4.6 generalizes [14, Theorem 4.4].

**Theorem 4.2.** *Assume that (6) holds and let  $M \in \mathfrak{M}_{P_1} \cap \mathfrak{M}_{P_2}$ .*

- (i) *If  $\mathcal{B}(S, \mathbb{R})$  is closed under  $M$  and if  $G \in \mathcal{B}(S, \mathbb{R})$  is admissible for  $M$ , then*

$$\mathfrak{sol}(E_G) = \{M_h + M_* \mid h \in \mathcal{B}(S, \mathbb{R})\}.$$

- (ii) *If  $G \in \mathcal{B}(S, \mathbb{R})$  is admissible for  $M$ , then*

$$\mathfrak{sol}(E_G) = \mathfrak{sol}(E_0) + M_*.$$

*Proof.* (i) Fix  $h \in \mathcal{B}(S, \mathbb{R})$ . From Lemmas 4.2 and 4.5 we have  $M_h + M_* \in \mathcal{M}(S, \mathbb{R}) + \mathcal{M}(S, \mathbb{R}) = \mathcal{M}(S, \mathbb{R})$  and  $M_h + M_* = \mathbf{T}M_h + \mathbf{T}M_* + G = \mathbf{T}(M_h + M_*) + G$ . Moreover, since  $\mathcal{B}(S, \mathbb{R})$  is closed under  $M$  and  $G \in \mathcal{B}(S, \mathbb{R})$  is admissible for  $M$ , we conclude that  $M_h + M_* \in \mathcal{B}(S, \mathbb{R}) + \mathcal{B}(S, \mathbb{R}) = \mathcal{B}(S, \mathbb{R})$ . Thus  $\{M_h + M_* \mid h \in \mathcal{B}(S, \mathbb{R})\} \subset \mathfrak{sol}(E_G)$ . The opposite inclusion follows from Lemma 4.4.

- (ii) It suffices to note that  $\mathfrak{sol}(E_0)$  is a subspace of  $\mathcal{B}(S, \mathbb{R})$ . □

*Remark 4.2.* The exemplary equation from Remark 4.1 shows that the admissibility assumption in assertion (ii) of Theorem 4.2 is necessary.

**Corollary 4.1.** *Assume that (6) holds and that  $\mathcal{B}(S, \mathbb{R})$  is closed under some  $M \in \mathfrak{M}_{P_1} \cap \mathfrak{M}_{P_2}$ . Then  $\mathfrak{sol}(E_G) \neq \emptyset$  if and only if  $G \in \mathcal{B}(S, \mathbb{R})$  is admissible for  $M$ .*

*Proof.* If  $G \in \mathcal{B}(S, \mathbb{R})$  is admissible for  $M \in \mathfrak{M}_{P_1} \cap \mathfrak{M}_{P_2}$ , then Lemma 4.5 yields  $M_* \in \mathfrak{sol}(E_G)$ .

If  $\mathfrak{sol}(E_G) \neq \emptyset$ , then Lemma 4.3 implies that the family  $\mathcal{G} \subset \mathcal{B}(S, \mathbb{R})$  is bounded, and hence  $G \in \mathcal{B}(S, \mathbb{R})$  and the function  $M_*$  is well defined for any

$M \in \mathfrak{B}$ . By Lemma 4.4 there exist  $M \in \mathfrak{M}_{P_1} \cap \mathfrak{M}_{P_2}$  and  $h \in \mathcal{B}(S, \mathbb{R})$  such that  $\varphi = M_h + M_*$ . Finally,  $M_* = \varphi - M_h \in \mathcal{B}(S, \mathbb{R}) - \mathcal{B}(S, \mathbb{R}) = \mathcal{B}(S, \mathbb{R})$ , which shows that  $G$  is admissible for  $M$ .  $\square$

We end this section showing how the described method works.

*Example 4.3.* Assume that  $G \in B([0, 1], \mathbb{R})$  is Riemann integrable and consider Riemann integrable solutions  $\varphi \in B([0, 1], \mathbb{R})$  of the equation

$$\varphi(x) = -\frac{1}{2}\varphi\left(\frac{x}{2}\right) - \frac{1}{2}\varphi\left(\frac{x+1}{2}\right) + G(x). \quad (12)$$

Clearly,  $\|\mathbf{T}\| = 1$  in the considered case. Fix  $h \in B([0, 1], \mathbb{R})$ ,  $M \in \mathfrak{B}$  and  $x \in [0, 1]$ . Easy calculations show that for every  $m \in \mathbb{N}$  we have

$$\mathbf{T}^m h(x) = (-1)^m \left[ \frac{1}{2^m} \sum_{i=0}^{2^m-1} h\left(\frac{x+i}{2^m}\right) \right]. \quad (13)$$

Then

$$\lim_{m \rightarrow \infty} \mathbf{T}^{2m} h(x) = \int_0^1 h(x) dx \quad \text{and} \quad \lim_{m \rightarrow \infty} \mathbf{T}^{2m-1} h(x) = -\int_0^1 h(x) dx,$$

and hence

$$M_h(x) = M((\mathbf{T}^m h(x))_{m \in \mathbb{N}}) = \frac{\int_0^1 h(x) dx - \int_0^1 h(x) dx}{2} = 0.$$

Since any Riemann integrable function is Borel measurable, it follows from Theorem 4.1, assertion (iv) of Example 4.1 and Lemma 4.3 that the trivial function is the only Riemann integrable solution of equation (12) with  $G = 0$ . Therefore, from assertion (ii) of Theorem 4.2 (cf. Lemma 4.3 and Corollary 4.1) we conclude that equation (12) has a Riemann integrable solution  $\varphi \in B([0, 1], \mathbb{R})$  if and only if the family  $\{\sum_{l=0}^{k-1} \mathbf{T}^l G \mid k \in \mathbb{N}\}$  is bounded, and in such a case, we have  $\varphi = M_*$ .

For example, if  $G$  is a constant function equals to  $c$ , then (13) yields

$$M_* = M\left(\left(\sum_{l=0}^{k-1} \mathbf{T}^l c\right)_{k \in \mathbb{N}}\right) = M\left(\left(\frac{c - (-c)^k}{2}\right)_{k \in \mathbb{N}}\right) = \frac{c}{2}.$$

But, if  $G$  is a characteristic functions of the set  $\mathbb{Q} \cap [0, 1]$ , then a similar calculation to that above leads to

$$M_*(x) = \begin{cases} \frac{1}{2}, & \text{if } x \in [0, 1] \cap \mathbb{Q}, \\ 0, & \text{if } x \in [0, 1] \setminus \mathbb{Q}. \end{cases}$$

## 5. The Case Where $\mathbf{T}$ is Expansive

In this section we want to get some information about existence of solutions of equation  $(E_G)$  in the case where  $\mathbf{T}$  is expansive. We first note that the method used in the previous section can not be applied if  $\|\mathbf{T}\| > 1$ . The reason is that in such a case functions  $M_h$  and  $M_*$  may not be defined correctly. To obtain the existence of a solution of equation  $(E_G)$  we may use one of fixed point theorems (see e.g. [1]). The below proposition is a simple application of the Schauder fixed point theorem (see [16]), but one can apply its generalization known as the Mönch fixed point theorem (see [13]).

**Proposition 5.1.** *Assume that  $\mathcal{Z} \subset \mathcal{B}(S, \mathbb{R})$  is a non-empty, closed and convex set such that  $\mathbf{T}(\mathcal{Z})$  is contained in a compact subset of  $\mathcal{Z}$ . Then Eq.  $(E_G)$  has a solution in  $\mathcal{Z}$ .*

An example of the set  $\mathcal{Z}$  satisfying assumptions of Proposition 5.1 is the set of all Lipschitzian functions with the Lipschitz constant  $L > 0$  provided that  $f$  is  $\mathcal{A}$ -measurable with respect to the second variable and there exists  $\alpha \leq 1$  such that  $\int_{\Omega} |g(\omega)| |f(x, \omega) - f(y, \omega)| d\mu(\omega) \leq \alpha |x - y|$  for all  $x, y \in S$ .

## 6. Special Subsets of $\mathbf{sol}(E_0)$ and $\mathbf{sol}(E_G)$

Basing on results from Sect. 4 we can extend all the main results obtained in [14] to larger classes of functions.

Assume that (6) holds. Fix a non-empty set  $S_0 \subset S$  and assume also that

$$f(x, \omega) = x \quad \text{for all } x \in S_0 \text{ and } \omega \in \Omega. \quad (14)$$

Given a family  $\mathcal{F} \subset \mathcal{B}(S, \mathbb{R})$  and a function  $\psi: S_0 \rightarrow \mathbb{R}$  we put

$$\mathcal{F}_{\psi} = \{h \in \mathcal{F} \mid h(x) = \psi(x) \text{ for every } x \in S_0\};$$

the symbol  $\mathcal{F}_0$  is used in the case where  $\psi(x) = 0$  for every  $x \in S_0$ . It is clear that  $\mathcal{B}(S, \mathbb{R})_0$  is a subspace of the space  $\mathcal{B}(S, \mathbb{R})$  and  $\mathcal{B}(S, \mathbb{R})$  is the union of all families  $\mathcal{B}(S, \mathbb{R})_{\psi}$ , where  $\psi$  runs over the set of all functions from  $S_0$  to  $\mathbb{R}$ . It is also easy to see that  $\mathcal{B}(S, \mathbb{R})_{\psi} + \mathcal{B}(S, \mathbb{R})_0 = \mathcal{B}(S, \mathbb{R})_{\psi}$ . In accordance with the notation introduced earlier, we have  $\mathbf{sol}(E_0)_{\psi} = \{\Phi \in \mathcal{B}(S, \mathbb{R})_{\psi} \mid \Phi = T\Phi\}$  and  $\mathbf{sol}(E_G)_{\psi} = \{\varphi \in \mathcal{B}(S, \mathbb{R})_{\psi} \mid \varphi = T\varphi + G\}$ .

Now, we want to determine  $\mathbf{sol}(E_0)_{\psi}$  and  $\mathbf{sol}(E_G)_{\psi}$  for any  $\psi: S_0 \rightarrow \mathbb{R}$ . For this purpose, we fix  $\psi: S_0 \rightarrow \mathbb{R}$  and put

$$\gamma = \int_{\Omega} g(\omega) d\mu(\omega).$$

Since the constant function 1 belongs to  $\mathcal{B}(S, \mathbb{R})$ , we see that (6) yields

$$|\gamma| = |\mathbf{T}1| \leq \|\mathbf{T}\| = 1.$$

Our first lemma is an immediate consequence of (14) and (4).

**Lemma 6.1.** *Assume that (14) holds.*

- (i) If  $h \in \mathcal{B}(S, \mathbb{R})_\psi$ , then  $\mathbf{T}h \in \mathcal{B}(S, \mathbb{R})_{\gamma\psi}$ .
- (ii) If  $\varphi \in \mathfrak{sol}(E_G)_\psi$ , then  $G \in \mathcal{B}(S, \mathbb{R})_{(1-\gamma)\psi}$ .

From assertion (i) of Lemma 6.1 we see that in the case where  $\gamma = 1$  each class  $\mathcal{I}_\psi$  is invariant under  $\mathbf{T}$ , and hence it makes sense to describe all the classes  $\mathfrak{sol}(E_0)_\psi$ . However, in the case where  $\gamma \neq 1$  we are interested only with describing the special class  $\mathfrak{sol}(E_0)_0$ , because the class  $\mathcal{I}_0$  is the only one that is invariant under  $\mathbf{T}$ .

**Lemma 6.2.** Assume that (6) and (14) hold. If  $M \in \mathfrak{M}_{P_1} \cap \mathfrak{M}_{P_2}$  and if  $h \in \mathcal{B}(S, \mathbb{R})_\psi$ , then  $M_h \in \mathcal{M}(S, \mathbb{R})_\psi$  in the case where  $\gamma = 1$  or  $M_h \in \mathcal{M}(S, \mathbb{R})_0$  in the case where  $\gamma \neq 1$ .

*Proof.* By Lemma 4.2 we have  $M_h \in \mathcal{M}(S, \mathbb{R})$ . Assume that  $h \in \mathcal{B}(S, \mathbb{R})_\psi$ . From assertion (i) of Lemma 6.1 we conclude that  $\mathbf{T}^m h \in \mathcal{B}(S, \mathbb{R})_{\gamma^m \psi}$  for every  $m \in \mathbb{N}$ . Fix  $x \in S_0$ . Then (14) implies that  $M_h(x) = M((\mathbf{T}^m h(x))_{m \in \mathbb{N}}) = M((\gamma^m \psi(x))_{m \in \mathbb{N}}) = \psi(x)M((\gamma^m)_{m \in \mathbb{N}})$ . Finally, it is enough to observe that  $M((\gamma^m)_{m \in \mathbb{N}}) = 1$  if  $\gamma = 1$  and that  $M((\gamma^m)_{m \in \mathbb{N}}) = 0$  if  $\gamma \in [-1, 1)$ .  $\square$

Combining Theorem 4.1 and Lemma 6.2 we obtain the following extension of [14, Theorem 3.2]; here and from now on we adopt the convention that  $\{B_h \mid h \in \mathcal{B}(S, \mathbb{R})_\psi\} = \emptyset$  provided that  $\mathcal{B}(S, \mathbb{R})_\psi = \emptyset$ .

**Theorem 6.1.** Assume that (6) and (14) hold. Let  $M \in \mathfrak{M}_{P_1} \cap \mathfrak{M}_{P_2}$ .

- (i) If  $\gamma = 1$  and if  $\mathcal{B}(S, \mathbb{R})_\psi$  is closed under  $M$ , then
 
$$\mathfrak{sol}(E_0)_\psi = \{M_h \mid h \in \mathcal{B}(S, \mathbb{R})_\psi\}.$$
- (ii) If  $\gamma \neq 1$  and if  $\mathcal{B}(S, \mathbb{R})_0$  is closed under  $M$ , then
 
$$\mathfrak{sol}(E_0)_0 = \{M_h \mid h \in \mathcal{B}(S, \mathbb{R})_0\}.$$
- (iii) If  $\gamma \neq 1$  and if  $\psi \neq 0$ , then

$$\mathfrak{sol}(E_0)_\psi = \emptyset.$$

To extend [14, Theorem 4.4] we need the following observation.

**Lemma 6.3.** Assume that the family  $\mathcal{G}$  is bounded and that (14) holds. If  $M \in \mathfrak{M}_{P_1} \cap \mathfrak{M}_{P_2}$  and if  $G \in \mathcal{B}(S, \mathbb{R})_{(1-\gamma)\psi}$ , then  $M_* \in \mathcal{M}(S, \mathbb{R})_0$  in the case where  $\gamma = 1$  or  $M_* \in \mathcal{M}(S, \mathbb{R})_\psi$  in the case where  $\gamma \neq 1$ .

*Proof.* Assume that  $M \in \mathfrak{M}_{P_1} \cap \mathfrak{M}_{P_2}$  and  $G \in \mathcal{B}(S, \mathbb{R})_{(1-\gamma)\psi}$ . By Lemma 4.5 we have  $M_* \in \mathcal{M}(S, \mathbb{R})$ . Fix  $x \in S_0$ . Assertion (i) of Lemma 6.1 and (14) give

$$M_*(x) = M \left( \left( \sum_{l=0}^{k-1} \mathbf{T}^l G(x) \right)_{k \in \mathbb{N}} \right) = \psi(x) M \left( \left( \sum_{l=0}^{k-1} (1-\gamma)\gamma^l \right)_{k \in \mathbb{N}} \right).$$

Finally, it is enough to observe that  $M((\sum_{l=0}^{k-1} (1-\gamma)\gamma^l)_{k \in \mathbb{N}}) = 0$  if  $\gamma = 1$  and that  $M((\sum_{l=0}^{k-1} (1-\gamma)\gamma^l)_{k \in \mathbb{N}}) = 1$  if  $\gamma \in [-1, 1)$ .  $\square$

**Theorem 6.2.** Assume that (6) and (14) hold. Let  $M \in \mathfrak{M}_{P_1} \cap \mathfrak{M}_{P_2}$  and let  $\gamma = 1$ .

- (i) If  $\mathcal{B}(S, \mathbb{R})_\psi$  is closed under  $M$  and if  $G \in \mathcal{B}(S, \mathbb{R})_0$  is admissible for  $M$ , then

$$\mathbf{sol}(E_G)_\psi = \{M_h + M_* \mid h \in \mathcal{B}(S, \mathbb{R})_\psi\}.$$

- (ii) If  $G \in \mathcal{B}(S, \mathbb{R})_0$  is admissible for  $M$ , then

$$\mathbf{sol}(E_G)_\psi = \mathbf{sol}(E_0)_\psi + M_*.$$

*Proof.* (i) It suffices to repeat the same arguments as in the proof of assertion (i) of Theorem 4.2 applying Lemma 6.3.

- (ii) Fix  $\varphi \in \mathbf{sol}(E_G)_\psi$ . Applying Lemmas 4.5 and 6.3 jointly with the admissibility of  $G$  we obtain  $\varphi - M_* \in \mathbf{sol}(E_0)_\psi$ . Then  $\varphi = (\varphi - M_*) + M_* \in \mathbf{sol}(E_0)_\psi + M_*$ . For the opposite inclusion, we fix  $\Phi \in \mathbf{sol}(E_0)_\psi$ . Using again Lemmas 4.5 and 6.3 jointly with the admissibility of  $G$  we get  $\Phi + M_* \in \mathbf{sol}(E_G)_\psi$ . □

The next result can be proved in the same way as Theorem 6.2, but it has no counterpart in [14].

**Theorem 6.3.** Assume that (6) and (14) hold. Let  $M \in \mathfrak{M}_{P_1} \cap \mathfrak{M}_{P_2}$  and let  $\gamma \neq 1$ .

- (i) If  $\mathcal{B}(S, \mathbb{R})_0$  is closed under  $M$  and if  $G \in \mathcal{B}(S, \mathbb{R})_{(1-\gamma)\psi}$  is admissible for  $M$ , then

$$\mathbf{sol}(E_G)_\psi = \{M_h + M_* \mid h \in \mathcal{B}(S, \mathbb{R})_0\}.$$

- (ii) If  $G \in \mathcal{B}(S, \mathbb{R})_{(1-\gamma)\psi}$  is admissible for  $M$ , then

$$\mathbf{sol}(E_G)_\psi = \mathbf{sol}(E_0)_0 + M_*.$$

*Remark 6.1.* The exemplary equation considered in Remark 4.1 shows that the admissibility assumption in assertion (ii) of Theorems 6.2 and 6.3 is necessary.

The next result is an immediate consequence of assertion (ii) of Theorems 6.2 and 6.3 and extends [14, Corollary 4.5].

**Corollary 6.1.** Assume that (6) and (14) hold. Let  $M \in \mathfrak{M}_{P_1} \cap \mathfrak{M}_{P_2}$ .

- (i) If  $\gamma = 1$ , then  $\mathbf{sol}(E_G)_\psi \neq \emptyset$  if and only if  $\mathbf{sol}(E_0)_\psi \neq \emptyset$  and  $G \in \mathcal{B}(S, \mathbb{R})_0$  is admissible for  $M$ .
- (ii) If  $\gamma \neq 1$ , then  $\mathbf{sol}(E_G)_\psi \neq \emptyset$  if and only if  $\mathbf{sol}(E_0)_\psi \neq \emptyset$  and  $G \in \mathcal{B}(S, \mathbb{R})_{(1-\gamma)\psi}$  is admissible for  $M$ .

Now, we can formulate many consequences of the main results of this paper about solutions of equations  $(E_0)$  and  $(E_G)$  in the classes of functions from example Example 2.1. However, we leave the details to the reader, who can consult [14, Section 5].



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